Position Feedback Nonlinear $\mathcal{H}_\infty$-Control for Inertia Wheel Pendulum Stabilization

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Paper received on 12/10/13, Accepted on 01/19/14.

Abstract. The paper deals with the position stabilization problem of the inertia wheel pendulum using a nonlinear $\mathcal{H}_\infty$ controller via position measurements for feedback. The main objective was to stabilize the pendulum at the up-right position in spite of the external disturbances. A local $\mathcal{H}_\infty$ controller is derived by means of a certain perturbation of the differential Riccati equations that appear while solving the corresponding $\mathcal{H}_\infty$-control problem for the linearized system. Since the initial conditions were far from the region of attraction of the desired equilibrium point, we construct a hybrid controller consisting of the swing-up control and the stabilization part. The performance of the proposed controller, applied to a perturbed academic wheel pendulum, was verified by simulations.

Keywords: Inertia wheel pendulum, nonlinear $\mathcal{H}_\infty$-control, output-feedback.

1 Introduction

The inertia wheel pendulum (IWP) is a nonlinear systems typically used to emulate postural sway such as bipedal walking motion, rocket thrust, the segway human transporter, and can be used to investigate research issues in control, autonomous navigation, group coordination, and other issues. One critical and interesting problem of the IWP is the stabilization of pendulum at the upright position because the open-loop equilibrium point is unstable and the closed-loop system can be sensitive to unmatched disturbances.

Nonlinear $\mathcal{H}_\infty$-control [1–3] is a tool to solve robust control problems and it has been successfully applied, among others, on the control of robot manipulators [4], underactuated mechanical systems [5], coil-power units [6], hard-disk drive servo systems [7], and hydraulic systems [8].

There are several local controllers design in the literature to stabilize the inertia wheel pendulum. For example, Ye *et al.* [9] proposed a backstepping technique for non-linear cascade systems whose driven subsystems have a feed-forward structure and include higher order terms. In their proposal, a small control is first assigned to stabilize the driven subsystem, and a simple backstepping procedure is then followed. López-Martínez *et al.* [10] considered a variable structure controller using a nonlinear sliding mode surface to deal with disturbances in the IWP. Recently, Turker *et al.* [11] developed an alternative stabilization procedure for a class of two degree-of-freedom under-actuated mechanical systems based on a set of transformations and a Lyapunov function.

The main objective of the paper is to design a $\mathcal{H}_\infty$ controller to stabilize the pendulum at the up-right position in spite of the external disturbances. Since the initial conditions were far from the region of attraction of the desired equilibrium point, we construct a hybrid controller consisting of the swing-up control and the stabilization part. For the swing-up control, it is proposed a desired periodic trajectory where a nonlinear $\mathcal{H}_\infty$-control for time-varying systems, taken from [3], drives the pendulum to the region of attraction of the desired equilibrium point. For the stabilization part, we linearize the model around that equilibrium point where a linear $\mathcal{H}_\infty$-control for autonomous systems, taken from [2], is applied to stabilize the pendulum.

The paper is organized as follows. The dynamic model and control objective are given in Section 2. Section 3 provides background material on output-feedback nonlinear $\mathcal{H}_\infty$-control for autonomous and non-autonomous systems. The nonlinear $\mathcal{H}_\infty$-control for swing-up and stabilization of the inertia wheel pendulum are designed in Section 4. Numerical simulations are presented in Section 5. Finally, conclusions are provided in Section 6.

2 Dynamic model

Dynamics of an inertia wheel pendulum, taken from [12], can be described as follows:

$$\underbrace{\begin{bmatrix} J_a + J_r & J_r \\ J_r & J_r \end{bmatrix}}_{J} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} \bar{m} g \sin(q_1) \\ 0 \end{bmatrix} = B(\tau + w_1) \tag{1}$$

where $q_1(t) \in \mathbb{R}$ is the angle of the pendulum, $q_2(t) \in \mathbb{R}$ is the angle of the disk (see Fig. 1), $\tau(t) \in \mathbb{R}$ is the control input, $t \in \mathbb{R}^+$ is the time, $w_1(t) \in \mathbb{R}$ is the unknown disturbance vector which is assumed square integrable over infinite-time (belongs to $\mathcal{L}_2$), J is the inertia matrix which is definite positive, and $B = [0,1]^T$. In the above equation of motion, $J_a = m_1 l_{c1}^2 + m_2 l_1^2 + J_p$ and $\bar{m} = m_1 l_{c1} + m_2 l_1$ where m_1 is the mass of the pendulum, m_2 is the mass of the wheel, l_1 is the length of the pendulum, l_{c1} is the distance to the center of mass of the pendulum, J_p is the moment of inertia of the pendulum, and J_r is the moment of inertia of the wheel.

The control objective pursued in this paper is to asymptotically stabilize the angle of the pendulum q_1 in the upper position set-point, that is,

$$\lim_{t \to \infty} \| q_1(t) - \pi \| = 0 \tag{2}$$

starting from the initial condition $q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$ in spite of the external disturbances $w_1(t) \in \mathbb{R}$. The position of the wheel and the pendulum are the only measurements available for feedback and these measurements are corrupted by the vector $w_y(t) \in \mathbb{R}^2$.

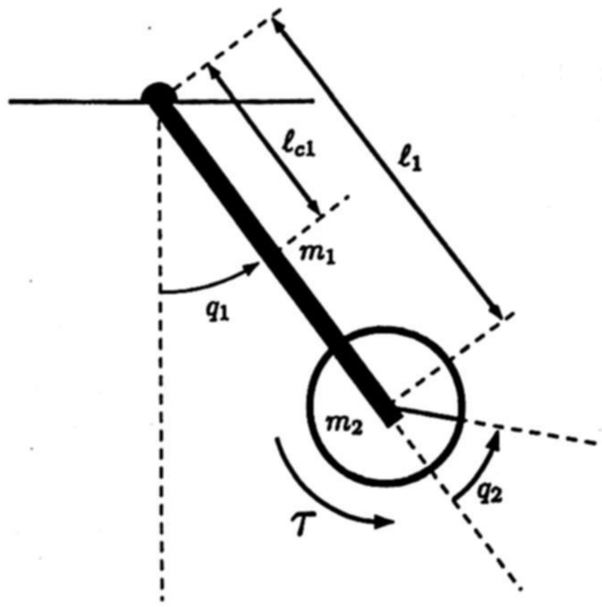

Fig. 1. Schematic representation of the inertia wheel pendulum.

3 Preliminaries

The present study focuses on a nonautonomous nonlinear system of the form

$$\dot{x} = f(x,t) + g_1(x,t)w + g_2(x,t)u \tag{3}$$
$$z = h_1(x,t) + k_{12}(x,t)u \tag{4}$$
$$y = h_2(x,t) + k_{21}(x,t)w \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the state-space vector, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^r$ is the unknown disturbance, $z(t) \in \mathbb{R}^l$ is the unknown output to be controlled, and $y(t) \in \mathbb{R}^p$ is the only available measurement on the system.

For the underlying system, the following assumptions are made throughout.

A1. The functions $f(x,t)$, $g_1(x,t)$, $g_2(x,t)$, $h_1(x,t)$, $h_2(x,t)$, $k_{12}(x,t)$, $k_{21}(x,t)$ are piecewise-continuous in t for all x and locally Lipschitz continuous in x for all t.
A2. $f(0,t) = 0$, $h_1(0,t) = 0$, and $h_2(0,t) = 0$ for all t.
A3.

$$h_1^T(x,t)k_{12}(x,t) = 0,\ k_{12}^T(x,t)k_{12}(x,t) = I$$
$$k_{21}(x,t)g_1^T(x,t) = 0,\ k_{21}(x,t)k_{21}^T(x,t) = I. \tag{6}$$

These assumptions are made for technical reasons (cf. [13]).

The $\mathcal{H}_\infty$-control problem in question is stated as follows. Given the system representation (3)–(5) and a real number $\gamma > 0$, it required to find (if any) a causal dynamic output-feedback compensator

$$u = \mathcal{K}(\xi,t), \qquad \dot{\xi} = \mathcal{F}(y,\xi) \tag{7}$$

with internal state $\xi \in \mathbb{R}^s$, such that the undisturbed closed-loop system is uniformly asymptotically stable around the origin and its $\mathcal{L}_2$ gain less than γ if the response z, resulting from w for initial state $x(t_0) = 0$ (and $\xi(t_0) = 0$), satisfies

$$\int_{t_0}^{t_1} \|z(t)\|^2 \, dt < \gamma^2 \int_{t_0}^{t_1} \|w(t)\|^2 \, dt \tag{8}$$

for all $t_1 > t_0$ and all piecewise-continuous functions $w(t) = [w_1 \; w_y]^T$. In turn, a locally admissible controller (7) constitutes a local solution of the $\mathcal{H}_\infty$-control problem if there exists a neighborhood U of the equilibrium such that inequality (8) is satisfied for all $t_1 > t_0$ and all piecewise-continuous functions $w(t)$ for which the state trajectory of the corresponding closed-loop system, starting from the initial point $x(t_0) = 0$ (and $\xi(t_0) = 0$), remains in U for all $t \in [t_0, t_1]$.

Under Assumptions A1-A3, coupled together, the corresponding Hamilton–Jacobi–Isaacs inequalities are subsequently linearized and a local solution of the $\mathcal{H}_\infty$-control problem is then obtained. The development involves the linear $\mathcal{H}_\infty$-control problem for the system

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u \\ z &= C_1(t)x + D_{12}(t)u \\ y &= C_2(t)x + D_{21}(t)w \end{aligned} \tag{9}$$

where

$$\begin{aligned} A(t) = \frac{\partial f_1}{\partial x}(0,t),\; B_1(t) = g_1(0,t),\; B_2(t) = g_2(0,t),\; C_1(t) = \frac{\partial h_1}{\partial x}(0,t), \\ C_2(t) = \frac{\partial h_2}{\partial x}(0,t),\; D_{12}(t) = k_{12}(0,t),\; D_{21}(t) = k_{21}(0,t). \end{aligned} \tag{10}$$

The following conditions are necessary and sufficient for a solution of the problem to exist:

C1. The equation

$$-\dot{P} = P(t)A(t) + A^T(t)P(t) + C_1^T(t)C_1(t) + P(t)[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T](t)P(t) \tag{11}$$

possesses a uniformly bounded positive semidefinite symmetric solution $P(t)$ such that the system

$$\dot{x} = [A - (B_2B_2^T - \gamma^{-2}B_1B_1^T)P](t)x(t) \tag{12}$$

is exponentially stable.

C2. Being specified with $A_1(t) = A(t) + \frac{1}{\gamma^2}B_1(t)B_1^T(t)P(t)$, the equation

$$\dot{Z} = A_1(t)Z(t) + Z(t)A_1^T(t) + B_1(t)B_1^T(t) + Z(t)[\frac{1}{\gamma^2}PB_2B_2^TP - C_2^TC_2](t)Z(t), \tag{13}$$

possesses a uniformly bounded positive semidefinite symmetric solution $Z(t)$, such that the system

$$\dot{x} = [A_1 - Z(C_2^TC_2 - \gamma^{-2}PB_2B_2^TP)](t)x(t) \tag{14}$$

is exponentially stable.

According to the time-varying strict bounded real lemma [14, p. 295], Conditions C1 and C2 ensure that there exists a positive constant ε_0 such that the system of the perturbed Riccati equations

$$-\dot{P}_\varepsilon = P_\varepsilon(t)A(t) + A^T(t)P_\varepsilon(t) + P_\varepsilon(t)[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T](t)P_\varepsilon(t) + C_1^T(t)C_1(t) + \varepsilon I, \tag{15}$$

$$\dot{Z}_\varepsilon = A_\varepsilon(t)Z_\varepsilon(t) + Z_\varepsilon(t)A_\varepsilon^T(t) + Z_\varepsilon(t)[\frac{1}{\gamma^2}P_\varepsilon B_2B_2^TP_\varepsilon - C_2^TC_2](t)Z_\varepsilon(t) + B_1(t)B_1^T(t) + \varepsilon I \tag{16}$$

has a unique uniformly bounded, positive definite symmetric solution $(P_\varepsilon(t), Z_\varepsilon(t))$ for each $\varepsilon \in (0, \varepsilon_0)$ where $A_\varepsilon(t) = A(t) + \frac{1}{\gamma^2}B_1(t)B_1^T(t)P_\varepsilon(t)$. Equations (15) and (16) are now utilized to derive a local solution of the $\mathcal{H}_\infty$-control problem for system (3)–(5).

Theorem 1. *Consider system (3)–(5) with Assumptions A1-A3. Let Conditions $C1$ and $C2$ be satisfied with a certain $\gamma > 0$ and let $(P_\varepsilon(t), Z_\varepsilon(t))$ be a uniformly bounded positive definite symmetric solution of (15), (16) under some $\varepsilon > 0$. Then, the causal dynamic output-feedback compensator*

$$\dot{\xi} = f(\xi,t) + [\frac{1}{\gamma^2}g_1(\xi,t)g_1^T(\xi,t) - g_2(\xi,t)g_2^T(\xi,t)]P_\varepsilon(t)\xi + Z_\varepsilon(t)C_2^T(t)[y(t) - h_2(\xi,t)], \tag{17}$$

$$u = -g_2^T(\xi,t)P_\varepsilon(t)\xi \tag{18}$$

is a local solution of the $\mathcal{H}_\infty$-control problem with the disturbance attenuation level γ.

In the autonomous case, the DREs (11), (13) degenerate to the algebraic Riccati equations by setting $\dot{P} = 0, \dot{Z} = 0$ and conditions $C1$ and $C2$ are simplified to

$C1''$. the equation

$$PA + A^TP + C_1^TC_1 + P[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T]P = 0 \tag{19}$$

possesses a positive semidefinite symmetric solution P such that the matrix $A - (B_2B_2^T - \gamma^{-2}B_1B_1^T)P$ has all eigenvalues with negative real part;

$C2''$. being specified with $A_1 = A + \frac{1}{\gamma^2}B_1B_1^TP$, the equation

$$A_1Z + ZA_1^T + B_1B_1^T + Z[\frac{1}{\gamma^2}PB_2B_2^TP - C_2^TC_2]Z = 0, \tag{20}$$

possesses a positive semidefinite symmetric solution Z such that the matrix $A_1 - Z(C_2^TC_2 - \gamma^{-2}PB_2B_2^TP)$ has all eigenvalues with negative real part.

Conditions $C1''$ and $C2''$ are known from [1] to be necessary and sufficient for a solution of the linear $\mathcal{H}_\infty$-control problem for the time-invariant version of system (9) to exist. According to the strict bounded real lemma, Conditions $C1''$ and $C2''$ ensure that there exists a positive constant ε_0 such that the system of the perturbed algebraic Riccati equations

$$P_\varepsilon A + A^TP_\varepsilon + C_1^TC_1 + P_\varepsilon[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T]P_\varepsilon + \varepsilon I = 0, \tag{21}$$

$$A_\varepsilon Z_\varepsilon + Z_\varepsilon A_\varepsilon^T + B_1B_1^T + Z_\varepsilon[\frac{1}{\gamma^2}P_\varepsilon B_2B_2^TP_\varepsilon - C_2^TC_2]Z_\varepsilon + \varepsilon I = 0 \tag{22}$$

has a unique positive definite symmetric solution $(P_\varepsilon, Z_\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$ where $A_\varepsilon = A + \frac{1}{\gamma^2}B_1B_1^TP_\varepsilon$. Based on this solution a time-invariant $\mathcal{H}_\infty$ controller is constructed as follows.

Theorem 2. *Let conditions $C1''$ and $C2''$ be satisfied for system (3)–(5) which is assumed to be time-invariant and let $(P_\varepsilon, Z_\varepsilon)$ be a positive definite symmetric solution of (21), (22) under some $\varepsilon > 0$. Then the time-invariant output-feedback*

$$\dot{\xi} = f(\xi) + \left[\frac{1}{\gamma^2}g_1(\xi)g_1^T(\xi) - g_2(\xi)g_2^T(\xi)\right]P_\varepsilon\xi + Z_\varepsilon C_2^T[y - h_2(\xi)], \tag{23}$$

$$u = -g_2^T(\xi)P_\varepsilon\xi \tag{24}$$

is a local solution of the $\mathcal{H}_\infty$-control problem in the autonomous case.

4 Nonlinear $\mathcal{H}_\infty$-control synthesis

4.1 Swing-up control

The role of the swing-up control is to drive the pendulum inside the region of attraction of the desired equilibrium point $[q_1^e q_2^e \dot{q}_1^e \dot{q}_2^e]^T = [\pi 000]^T$. For this purpose, let us follow the line of reasoning of Orlov *et al.* [15] where the modified Van der Pol oscillator

$$\ddot{q}_d + \alpha\left[\left(q_d^2 + \frac{\dot{q}_d^2}{\mu^2}\right) - \rho_v^2\right]\dot{q}_d + \mu^2 q_d = 0, \qquad \alpha, \rho_v, \mu > 0 \tag{25}$$

was used to generate a periodic reference trajectory for the pendulum. Here, $q_d(t) \in \mathbb{R}$ stand for the desired periodic trajectory, the parameter ρ_v controls the amplitude of the limit cycle, μ control its frequency, and α controls the speed of the limit cycle transient.

For the synthesis of the $\mathcal{H}_\infty$ tracking controller, consider the state-space vector $x = [x_1\ x_3]^T$ where $x_1 = q_1 - q_d$ and $x_3 = \dot{q}_1 - \dot{q}_d$. Then (1), represented in terms of the state-space vector, takes the form

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_3 &= -J_a^{-1} h \sin(x_1 + q_d) - \ddot{q}_d - J_a^{-1} w_1 - J_a^{-1} \tau, \end{aligned} \tag{26}$$

where $h = \bar{m}g$. It should be pointed out that it is assumed that velocity of the wheel $\dot{q}_2$ remains bounded for all $t \in [0, t_s)$ where t_s is the transition instant between the swing-up controller and the stabilizing controller. The objective is to design a controller of the form

$$\tau = -J_a(\ddot{q}_d + J_a^{-1} h \sin(q_1) + u) \tag{27}$$

that imposes on the disturbance-free manipulator motion desired stability properties around $q_d(t)$ while also locally attenuating the effect of the disturbances. Thus, the controller to be constructed consists of the trajectory feedforward compensator and a disturbance attenuator $u(t)$; internally stabilizing the closed-loop system around the desired trajectory.

In the sequel, we confine our design objective to position-control where

1. The output to be controlled is given by

$$z = \begin{bmatrix} u \\ \rho x_1 \end{bmatrix} \tag{28}$$

with a positive weight coefficient ρ, and

2. The position of the wheel and the pendulum are the only measurements available for feedback and these measurements are corrupted by the vector $w_y(t) \in \mathbb{R}^2$, that is,

$$y = \begin{bmatrix} x_1 + q_d \\ x_2 \end{bmatrix} + w_y(t). \tag{29}$$

The system (26)–(29) can be specified as in (3)–(5) with

$$\begin{aligned} f(x) &= \begin{bmatrix} x_3 \\ 0 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 & 0\ 0 \\ -J_a^{-1} & 0\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ -J_a^{-1} \end{bmatrix}, \\ h_1(x) &= \begin{bmatrix} 0 \\ \rho x_1 \end{bmatrix}, \quad k_{12}(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h_2(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad k_{21}(x) = \begin{bmatrix} 0_{2\times 1}\ I_2 \end{bmatrix}. \end{aligned} \tag{30}$$

Hereinafter, I_n and $0_{n\times m}$ stand for the $n \times n$ identity matrix and the $n \times m$ matrix of zeros, respectively. Thus, a solution to the $\mathcal{H}_\infty$ output tracking controller synthesis involves the standard linear $\mathcal{H}_\infty$-control problem for the nonautonomous linearized system (9) where matrices (10) are explicitly given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ -J_a^{-1} & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -J_a^{-1} \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} I_2 \; 0_{2\times 2} \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0_{2\times 1} \; I_2 \end{bmatrix}. \tag{31}$$

Finally, by applying Theorem 1 subject to (30) and (31), the $\mathcal{H}_\infty$-control problem is solved.

4.2 Stabilization control

For this stage, the nonlinear system (1) will be linearized around the desired equilibrium point therefore, the control input τ will be injected from the $\mathcal{H}_\infty$-control (24) without nonlinear compensation terms, that is, $\tau = u$. Now, consider the state-space vector $x = [x_1 \; x_2 \; x_3 \; x_4]^T$ where $x_1 = q_1 - \pi$, $x_2 = q_2$, $x_3 = \dot{q}_1$, and $x_4 = \dot{q}_2$. Then (1), represented in terms of the state-space vector, takes the form

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -J_a^{-1} h \sin(x_1 + \pi) - J_a^{-1} w_1 - J_a^{-1} u \\ \dot{x}_4 &= J_a^{-1} h \sin(x_1 + \pi) + (J_r^{-1} + J_a^{-1}) w_1 + (J_r^{-1} + J_a^{-1}) u. \end{aligned} \tag{32}$$

We confine our design objective to position regulation where the output to be controlled is given by

$$z = \begin{bmatrix} u \\ \rho x_1 \\ \rho x_2 \end{bmatrix}. \tag{33}$$

The system (29), (32)–(33) can be specified as in (3)–(5) with

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ -J_a^{-1} h \sin(x_1 + \pi) \\ J_a^{-1} h \sin(x_1 + \pi) \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0_{2\times 1} & 0_{2\times 2} \\ J^{-1}B & 0_{2\times 2} \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0_{2\times 1} \\ J^{-1}B \end{bmatrix},$$
$$h_1(x) = \begin{bmatrix} 0 \\ \rho x_1 \\ \rho x_2 \end{bmatrix}, \quad k_{12}(x) = \begin{bmatrix} 1 \\ 0_{2\times 1} \end{bmatrix}, \quad h_2(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad k_{21}(x) = \begin{bmatrix} 0_{2\times 1} \; I_2 \end{bmatrix}. \tag{34}$$

Thus, a solution to the $\mathcal{H}_\infty$ output regulator synthesis involves the standard linear $\mathcal{H}_\infty$-control problem for the autonomous linearized system (9) where matrices (10) are explicitly given by

$$A = \begin{bmatrix} 0 & 0\,1\,0 \\ 0 & 0\,0\,1 \\ J_a^{-1}h & 0\,0\,0 \\ -J_a^{-1}h & 0\,0\,0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0_{2\times1} & 0_{2\times2} \\ J^{-1}B & 0_{2\times2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0_{2\times1} \\ J^{-1}B \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0_{1\times2} & 0_{1\times2} \\ \rho I_2 & 0_{2\times2} \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 \\ 0_{2\times1} \end{bmatrix}, \quad C_2 = \begin{bmatrix} I_2 & 0_{2\times1} \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0_{2\times1} & I_2 \end{bmatrix}. \tag{35}$$

Finally, by applying Theorem 2 subject to (34) and (35), the $\mathcal{H}_\infty$-control problem is solved.

5 Simulation results

Forthcoming result were based on the laboratory inertial wheel pendulum from Mechatronics Control Kit, prototype manufactured by QUANSER Inc., where $J_a = 4.572 \times 10^{-3}$, $J_r = 2.495 \times 10^{-5}$, $h = 0.3544$. As was specified in Section 2, the position initial conditions for the IWP were set to $q_1(0) = q_2(0) = 0$ [rad] whereas all the velocity initial conditions were set to $\dot{q}_1(0) = \dot{q}_2(t) = 0$ [rad/s].

The parameters specified for the modified Van der Pol equation (25), to generate a periodic trajectory with amplitude $|q_d| = 3\pi/4$ [rad], were $\alpha = 100$, $\rho_v = 3\pi/4$, and $\mu = 4$ (see Fig. 2). The parameters $\rho = 250$, $\gamma = 10$, and $\varepsilon = 5$ were chosen for the swing-up $\mathcal{H}_\infty$ controller and for $\mathcal{H}_\infty$ regulator were $\rho = 1$, $\gamma = 40$, and $\varepsilon = 0$.

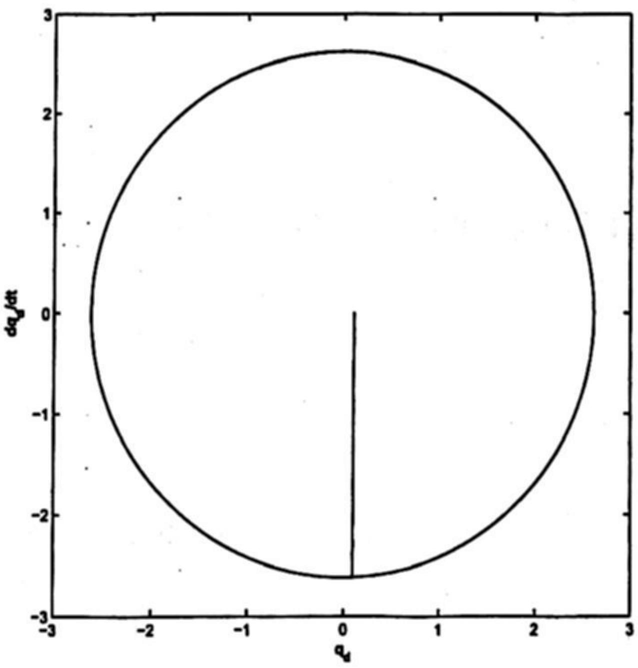

Fig. 2. Phase portrait produced by the modified Van der Pol equation with parameters $\alpha = 100$, $\rho_v = 3\pi/4$, and $\mu = 4$ initialized at $q_d(0) = 3\pi/4$ and $\dot{q}_d(0) = 0$.

Figure 3 provides the position, velocity, and torque of the pendulum without disturbances. Figure 4 shows that positions and velocities of the inertia wheel pendulum are driven to the desired equilibrium point in spite of the external disturbances

$$w_1 = 0.1\cos(2t), \quad w_y = \left[1 \times 10^{-3}\cos(5t)\ 2 \times 10^{-3}\cos(3t)\right]^T. \qquad (36)$$

The switching between controllers occur at $t_s = 0.4$ [s]. There were no significant differences between Figs. 3 and 4 in terms of the overshoot. As can be seen, also, from these Figures, is that the velocity $\dot{q}_2(t)$ does not escape to infinity in finite time t_s, however, high velocity of the wheel is required to satisfy the control objective. New methods for swing-up the pendulum without demanding high energy must be investigated.

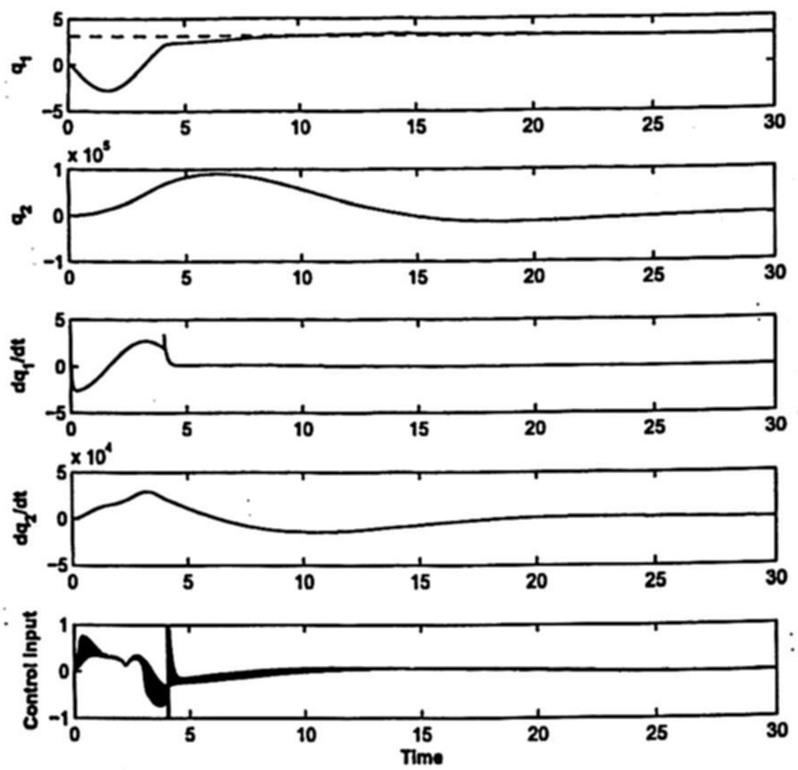

Fig. 3. Time responses of the closed-loop system.

6 Conclusions

In this paper we propose a nonlinear $\mathcal{H}_\infty$-controller to solve the stabilization problem, at the upright position, of a inertia wheel pendulum operating under uncertain conditions assuming position feedback only. The proposed controller consists of the swing-up part and the stabilization part. For the swing-up control, we proposed a Van-der-Pol equation to generate a continuously differentiable desired periodic trajectory where the $\mathcal{H}_\infty$-control successfully drives the pendulum to the region of attraction of the desired equilibrium point. Finally, the nonlinear $\mathcal{H}_\infty$ regulator stabilizes the pendulum at the

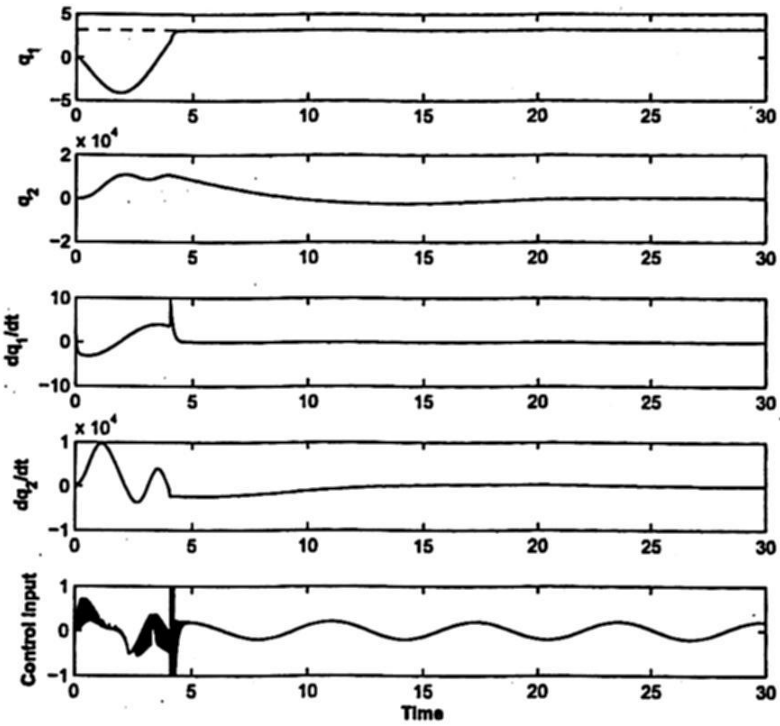

Fig. 4. Time responses of the closed-loop system.

desired position in spite of the presence of external disturbances, as was expected. The most important limitation lies in the swing-up control part where boundedness of the velocity of the wheel must be investigated.

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